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A π^1_2 SINGLETON WITH NO SHARP IN A GENERIC EXTENSION OF $L^{\#}$

BY

RENÉ DAVID

ABSTRACT

We may add to the minimal model with all the sharps for reals a c.c.c. generic Δ^1_3 real with *no* sharp.

In this paper we show that if we assume the consistency of the theory: $\mathrm{ZF}+\forall x(x \subset \omega \rightarrow x^{\#}$ exists) there is a model of this theory and a c.c.c. generic extension N of it, in which there is a π^1_2 real singleton a such that each element of N is constructible from a and so that $a^{\#}$ does not exist in N.

We obtain the first model M by considering in some model of $\mathrm{ZF}^{\#}$ (we note $\mathrm{ZF}^{\#}$ the theory: $\mathrm{ZF}+\forall x(x \subset \omega \rightarrow x^{\#}$ exists)) the collection $L^{\#}$ constructed by saturating L by the sharp operation on the reals.

Our proof then uses the methods developed in [1], [3], [4] for constructing a π^1_2 singleton with good properties.

Before giving our main result, let us give a precise definition of the collection $L^{\#}$.

PROPOSITION I–1. *Let M be a model of* $\mathrm{ZF}^{\#}$. *There exist an ordinal ξ and a family $(a_\alpha, 0_\alpha)_{\alpha<\xi}$ unique such that*:

(1) *for all α in ξ*: $a_\alpha \subset \omega$, $0_\alpha \subset \omega . \alpha$;

(2) *for all α in ξ*: *if* $\alpha = \bigcup \alpha$, $0_\alpha = \bigcup_{\beta<\alpha} 0_\beta$,

if $\alpha = \beta + 1$, $0_\alpha = 0_\beta \cup \{\omega . \beta + n / n \in a^{\#}_\beta\}$;

(3) *for all α in ξ*: *α is countable in $L(0_\alpha)$ and a_α is the first real x, in the canonical order of $L(0_\alpha)$, such that $L(x) = L(0_\alpha)$.*

(4) *ξ is not countable in* $L(\bigcup_{\alpha<\xi} 0_\alpha)$.

PROOF. Properties (1), (2), (3) define a_α and 0_α by induction. ξ is the first ordinal α such that α is not countable in $L(0_\alpha)$.

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DEFINITION I–2. Let M be a model of $\mathrm{ZF}^{\#}$ and $A = \bigcup_{\alpha<\xi} 0_\alpha$. The collection $L^{\#}$ is defined by the formula: $x \in L^{\#} \leftrightarrow x \in L(A)$.

Our main theorem is now

THEOREM I. *There is a* π^1_2 *formula* $\varphi(x)$, *where* x *is real, such that the following are provable in* ZF:

(1) $V = L^{\#} \rightarrow \neg \exists x\, \varphi(x)$.

(2) $\aleph_1 = \aleph_1^{L^{\#}} \rightarrow \exists^{\leq 1} x\, \varphi(x)$.

(3) *If* $\mathrm{ZF}^{\#}$ *is consistent, then so is*

$$\mathrm{ZFC} + \mathrm{GCH} + \aleph_1 = \aleph_1^{L^{\#}} + \exists a\, (\varphi(a)\ \&\ V = L(a) = L^{\#}(a)).$$

REMARKS. (1) This theorem shows that the following result, due to R. Jensen, is in some sense the best possible:

If N is an extension of $L^{\#}$ which satisfies: $\forall x (x \subset \omega \rightarrow x^{\#}$ exists$) + \neg 0^{\dagger} + L^{\#} = K$ (where K is the core model of Dodd–Jensen), then $L^{\#}$ is Σ^1_3 absolute in N.

The model which will prove part (3) of the theorem gives a counterexample:

N satisfies $\exists x\ (\varphi(x)\ \&\ x^{\#}$ does not exist$)$ whereas $M = (L^{\#})^N$ satisfies $\neg \exists x\, \varphi(x)$.

(2) At the end of this paper we give a sketch of the proof of a theorem which shows that we may add a π^1_3 real singleton generically to $L^{\#}$ and preserve the sharp property for all the reals. The proof is very near that of [2].

I. Some properties of $L^{\#}$

PROPOSITION I–3. *The collection* $L^{\#}$ *satisfies*:

(1) $\mathrm{ZFC} + \mathrm{GCH} + \exists X \subset \aleph_1,\ V = L(X)$,

(2) $\forall x (x \subset \omega \rightarrow x^{\#}$ *exists*$) + V = L^{\#}$,

(3) $\forall x\, (x \subset \omega \rightarrow \exists \alpha < \aleph_1,\ x \in L(a_\alpha))$.

PROOF. It is clear that the ordinal ξ defined in Proposition I–1 is such that $\xi = \aleph_1^{L^{\#}}$. (1) follows. (2) and (3) follow from the following lemma and the fact that $0_\alpha = A \cap \omega \cdot \alpha$ and $L(0_\alpha) = L(a_\alpha)$.

LEMMA I–4. *If* a *and* b *are reals,* $a^{\#}$ *exists and* b *is constructible from* a, *then* $b^{\#}$ *exists.*

PROPOSITION I–5. *In every model of* $\mathrm{ZF} + V = L^{\#}$ *there exists a* Δ^1_3 *well-ordering of* R.

PROOF. Let $\varphi(x, y)$ be the formula:

$x, y \subset \omega$ and $\exists \alpha < \aleph_1$ [$x \in L(a_\alpha)$ and $\forall \beta < \alpha$ $x \notin L(a_\beta)$ and
$((\exists \beta < \alpha$ $y \in L(a_\beta))$ or ($y \in L(a_\alpha)$ and y is before x
in the canonical well-ordeirng of $L(a_\alpha))))$].

φ clearly defines a well-ordering of $R \cap L^{\#}$.

It is easily seen that "$x = a_\alpha$" is a Σ_2^{ZF} formula, and so φ is equivalent to a Σ_4^1 formula. We have to remove one quantifier.

Let $\psi(x, y)$ be the formula:

$$\exists \alpha, \beta < \aleph_1 \ (L_\alpha(a_\beta) \models \mathrm{ZFC}^- \ \& \ x, y \in L_\alpha(a_\beta)$$

$$\& \ L_\alpha(a_\beta) \models \varphi(x, y)).$$

ψ is equivalent to a Σ_3^1 formula. The fact that φ and ψ are equivalent follows from Lemma I–6:

LEMMA I–6. *If $L_\alpha(a^{\#}) \models \mathrm{ZFC}^-$ then α is inaccessible in $L(a)$ and so $R \cap L(a) \subset L_\alpha(a^{\#})$.*

II. A c.c.c. generic extension of a model of $V = L^{\#}$ which is not closed under the sharp operation

PROPOSITION II–1. *Let M be a model of $\mathrm{ZF} + V = L^{\#}$. There is a unique sequence of reals $(r_\beta)_{\beta < \aleph_1}$ such that:*

$\forall \alpha < \aleph_1$ $\forall \beta (\omega . \alpha \leqq \beta < \omega . (\alpha + 1) \rightarrow r_\beta$ is the first real in the canonical order of $L(0_\alpha)$ which is not in the set $\{r_\gamma \mid \gamma < \beta\}$).

PROOF. The sequence is defined by induction. Since for each α in $\aleph_1$, α is countable in $L(0_\alpha)$, we only have to see that for all α in $\aleph_1$ the sequence $(r_\beta)_{\beta < \omega \cdot (\alpha+1)}$ is in $L(0_\alpha)$, but this is also done by induction.

DEFINITION II–2. Let M be a model of $\mathrm{ZF} + V = L^{\#}$ and $A = \bigcup_{\alpha < \aleph_1} 0_\alpha$. We define the set P_1 of forcing conditions by:

$$s \in P_1 \leftrightarrow s = (s(0), s(1)) \in P_f(\omega) \times P_f(A)$$

$$(\text{where } P_f(X) = \{x \subset X / x \text{ is finite}\}),$$

$$s \leqq s' \leftrightarrow s(i) \supset s'(i), \ i = 0, 1 \ \& \ \forall \alpha \in s'(1) \ S(r_\alpha) \cap s(0) \subset s'(0)$$

$$(\text{where } S(x) = \{x \restriction n / n \in \omega\}).$$

PROPOSITION II–3. (1) *P_1 satisfies the $\aleph_1$ chain condition.*

(2) *If G is a M generic over P_1 and $g = \{n / \exists s \in G,\ n \in s(0)\}$ then $M(G) = M(g)$ and $\forall \alpha < \aleph_1$ ($\alpha \in A \leftrightarrow S(r_\alpha) \cap g$ is finite).*

PROOF. See [1].

PROPOSITION II–4. $\forall \alpha < \aleph_1$, $0_\alpha \in L(g)$.

PROOF. By induction.

$\alpha = 0$: trivial.

$\alpha = \beta + 1$: by induction $0_\beta \in L(g)$ but $\forall \gamma < \omega . \alpha$ ($\gamma \in 0_\alpha \leftrightarrow S(r_\gamma) \cap g$ is finite) and $(r_\gamma)_{\gamma < \omega . \alpha} \in L(0_\beta)$; so $0_\alpha \in L(g)$.

$\alpha = \bigcup \alpha$: by induction $\forall \beta < \alpha$, $0_\beta \in L(g)$. Since $0_\alpha = \bigcup_{\beta < \alpha} 0_\beta$ we only have to see that the sequence $(0_\beta)_{\beta < \alpha}$ is an element of $L(g)$; this is because $L(g)$ satisfies: $\forall \beta < \alpha$, 0_β exists, and absoluteness of the construction of $(0_\beta)_{\beta < \alpha}$.

COROLLARY. *$M(g)$ satisfies: $V = L(g) = L^{\#}(g)$ and $g^{\#}$ does not exist.*

PROPOSITION II–5. *There is a Σ_1^{ZF} formula $\theta(x, \alpha, y)$ such that if M is a model of $\mathrm{ZF} + V = L^{\#}$ and g is a M generic over P_1 then $M(g)$ satisfies:*

$$\forall \alpha < \aleph_1, \forall x (x = 0_\alpha \leftrightarrow \theta(x, \alpha, g)).$$

PROOF. It is easy to find a Σ_1^{ZF} formula $E(x, \alpha, y)$ such that:

$$\begin{aligned} \mathrm{ZF} \vdash \forall y \subset \mathrm{ORD} (\forall x, \alpha (E(x, \alpha, y) \rightarrow x \subset \omega \ \&\ \alpha < \aleph_1^{L(y)}) \ \& \\ \forall x (x \subset \omega \ \&\ x \in L(y) \rightarrow \exists! \alpha E(x, \alpha, y)) \ \& \\ \forall \alpha < \aleph_1^{L(y)} (\exists! x E(x, \alpha, y) \ \&\ x \in L(y)). \end{aligned}$$

(E gives a uniform enumeration of the reals in $L(y)$.)

We define θ by:

$\alpha < \aleph_1$ and $\exists f, g$ functions with domain $\alpha + 1$ for f and $\omega . (\alpha + 1)$ for g such that $x = f(\alpha)$ and

$$\forall \beta \leqq \alpha \left(\beta = \bigcup \beta \rightarrow f(\beta) = \bigcup_{\gamma < \beta} f(\gamma) \right)$$

and

$$\begin{aligned} &\forall \beta < \alpha \ \forall \gamma < \omega . (\beta + 1) \exists z \, [z = g(\gamma) \ \& \\ &(\gamma \in f(\beta + 1) \leftrightarrow S(z) \cap y \text{ is finite}) \ \& \\ &\exists \lambda (E(z, \lambda, f(\beta)) \ \& \\ &\forall \mu < \lambda \ \exists \eta < \gamma E(g(\eta), \mu, f(\beta)))]. \end{aligned}$$

It is clear that θ is a Σ_1^{ZF} formula. In this formula $f(\beta)$ is 0_β and $g(\gamma)$ is r_γ. θ has the good properties because of Proposition II–3.

COROLLARY II–6. *In $M(g)$ the formula "$x \in R \cap L^\#$" is equivalent to a $\Sigma_1^{ZF}(x, g)$ formula.*

$$x \in R \cap L^\# \leftrightarrow x \subset \omega \ \& \ \exists \alpha, \beta < \aleph_1, \ x \in L_\alpha(0_\beta).$$

III. Proof of Theorem I

Let M be a model of ZF + $V = L^\#$. Following the construction in [3] we show that there is a sequence $(T_n, f_n, \tau_n)_{n\in\omega}$ such that:

(1) $\forall n \ T_n$ is a normal tree of length $\aleph_1$;

(2) $\forall n \geqq 1 \ f_n : T_n \to T_{n-1}$ and

$\forall x \in T_n \ |f_n(x)| = |x| + 1$ ($|\cdot|$ is the length function in a tree);

for each $\aleph_1$-branch β in T_{n-1}, $\{x \in T_n \mid f_n(x) \in \beta\}$ is a normal subtree of length $\aleph_1$ in T_n;

(3) $\forall n \in \omega \ \tau_n : T_n \to \omega$ and

$\forall x, y \in T_n (x \leqq_{T_n} y \to \tau_n(x) = \tau_n(y) > \tau_{n-1}(f_n(x)))$.

$\forall y \, \forall m (y \in T_{n-1} \ \& \ |y| = 0 \ \& \ m \in \omega \ \& \ m > \tau_{n-1}(y) \to \exists x \in T_n(|x| = 0 \ \& \ \tau_n(x) = m \ \& \ f_n(x) >_{T_{n-1}} y))$;

(4) if N is an extension of M which preserves $\aleph_1$, then there is for each n at most one branch of length $\aleph_1$ in T_n;

(5) let $P_2 = \{p \mid \operatorname{dom} p = n \ \ \& \ \ \forall i < n \ \ p(i) \in T_i \ \ \& \ \ \forall i < n-1 \ \ p(i) \geqq f_{i+1}(p(i+1))\}$;

the order on P_2 is

$$p \leqq q \leftrightarrow \operatorname{dom} p \supset \operatorname{dom} q \ \& \ \forall i \in \operatorname{dom} q \ p(i) \geqq_{T_i} q(i).$$

Then P_2 satisfies the $\aleph_1$ chain condition.

PROOF. The construction follows that of [3]. At limit level it is as follows:

Let $h : \aleph_1 \to \aleph_1$ be the function defined by:

$$h(\alpha) \text{ is the first ordinal } \beta \text{ such that } \alpha \text{ is countable in } L(0_\beta).$$

(We know that $\forall \alpha \ h(\alpha) \leqq \alpha$.)

$T_n \restriction \omega . \alpha + 1$ is constructed by induction on n.

$n = 0$: Let $\beta = h(\alpha)$ and $\eta_{0,\alpha}$ be the first ordinal η such that:

$$T_0 \restriction \omega . \alpha \in L_\eta(0_\beta),$$

$$L_\eta(0_\beta) \models \text{ZFC}^- \ \& \ \alpha \text{ is countable.}$$

$T_0 \upharpoonright \omega \cdot \alpha + 1$ is constructed by taking a $L_{\eta_{0,\alpha}}(0_\beta)$ generic over the same set of forcing conditions as in [3]; this generic is in $L(0_\beta)$.

$n+1$: The construction is the same but $\eta_{n+1,\alpha}$ is the first η such that $T_n \upharpoonright \omega \cdot \alpha + 1$, $T_{n+1} \upharpoonright \omega \cdot \alpha \in L_\eta(0_\beta)$ and $L_\eta(0_\beta) \models \mathrm{ZFC}^-$ & α is countable.

To see that this construction is possible we have to show by induction that:

$$\forall n \in \omega \, \forall \alpha < \aleph_1 \; T_n \upharpoonright \omega . \alpha \in L(0_{h(\alpha)}).$$

The proof for $n+1$ is the same as that of $n=0$. We prove by induction that

$$T_0 \upharpoonright \omega . \alpha \in L(0_{h(\alpha)}).$$

$\alpha = \gamma + 1$: Let $\beta = h(\gamma) = h(\alpha)$. $T_0 \upharpoonright \omega . \gamma \in L(0_\beta)$ and so $T_0 \upharpoonright \omega . \gamma + 1 \in L(0_\beta)$ but the construction for successor case is trivial so $T_0 \upharpoonright \omega . \alpha \in L(0_\beta)$.

$\alpha = \bigcup \alpha$: $T_0 \upharpoonright \omega . \alpha = \bigcup_{\gamma<\alpha} T_0 \upharpoonright \omega . \gamma$ but $\forall \gamma < \alpha \quad h(\gamma) \leqq h(\alpha)$ and $T_0 \upharpoonright \omega . \gamma \in L(0_{h(\gamma)})$, the construction being absolute, we have $(T_0 \upharpoonright \omega . \gamma)_{\gamma<\alpha} \in L(0_{h(\alpha)})$ and so $T_0 \upharpoonright \omega . \alpha \in L(0_{h(\alpha)})$.
The properties (1), (2), (3), (4) are proved just as in [3].

Let us now show that P_2 satisfies the $\aleph_1$ chain condition.

We show as in [3] that for each $n \in \omega$ T^*_n is a Souslin tree in M_n. This is done by induction.

Let T be T^*_{n+1} and $B \subset \aleph_1$ be a code for the branch in T^*_n such that M_{n+1} is $L(A, B)$. (Recall that $A = \bigcup_{\alpha<\aleph_1} 0_\alpha$.)

Let $C \subset T$ be a maximal antichain. We have to show that C is countable in M_{n+1}. Let X be a countable elementary substructure of $L_{\aleph_2}(A, B)$ such that A, B, C, T are in X. There is a unique isomorphism π

$$X \underset{\pi}{\xrightarrow{\simeq}} L_\beta(A \cap \alpha, B \cap \alpha), \qquad \text{where } \alpha = \aleph_1 \cap X = \pi(\aleph_1) < \aleph_1.$$

We may suppose that $h(\alpha) = \alpha$ (since we may suppose that for γ in $\aleph_1 \cap X$, $\aleph_1^{L(0_\gamma)}$ also is in X).

It is then sufficient to prove, as in [3], that

$$L_\beta(A \cap \alpha, B \cap \alpha) \subset L_{\eta_{n+1,\alpha}}(0_{h(\alpha)})$$

but: $A \cap \alpha = 0_\alpha$ (since $\omega . \alpha = \alpha$); $h(\alpha) = \alpha$; $B \cap \alpha \in L_{\eta_{n+1,\alpha}}(0_\alpha)$ (since by the construction of T, $T_n \upharpoonright \omega \cdot \alpha + 1$ is in $L_{\eta_{n+1,\alpha}}(0_\alpha)$); α is not countable in $L_\beta(0_\alpha)$ and so $\beta < \eta_{n+1,\alpha}$. So we are done, and the existence of the sequence (T_n, f_n, τ_n) is proved.

Definition III–1. We define the ordered set of conditions P by:

$$q \in P \leftrightarrow q = (s,p) \;\&\; s \in P_1 \;\&\; p \in P_2 \;\&\; s(0) = \tau(p)$$

$$(\text{where } \tau(p) = \{\tau_i(p(i)) \mid i \in \operatorname{dom} p\}),$$

$$(s,p) \leqq (s',p') \leftrightarrow s \leqq s' \;\&\; p \leqq p'.$$

PROPOSITION III–2. *P satisfies the $\aleph_1$ chain condition.*

PROOF. We first show that if $q = (s,p)$, $q' = (s',p')$ and $q, q' \in P$ and $\tau(p) = \tau(p')$ then q and q' are compatible if and only if p and p' are compatible. Let $p_1 \leqq p$, p'; we may suppose that $\tau(p_1) = \tau(p) \cup \tau(p')$; then $q_1 = (s_1, p_1)$ extends both q and q', where s_1 is defined by: $s_1(0) = s(0) = s'(0)$ and $s_1(1) = s(1) \cup s'(1)$.

Suppose then that $(q_\alpha)_{\alpha<\aleph_1}$ is an antichain in P. Since for each α $\tau(p_\alpha)$ is a finite subset of ω we may suppose that there is a finite subset x of ω such that for all α $\tau(p_\alpha) = s_\alpha(0) = x$; but then for all α, β s_α and s_β are compatible and so $(p_\alpha)_{\alpha<\aleph_1}$ is an antichain in P_2 and this is impossible.

PROPOSITION III–3. *Let G be a M generic over P and $g = \{n \in \omega \mid \exists q = (s,p) \in G,\ n \in s(0) = \sigma(p)\}$. Then $M(G) = M(g)$ and $M(g)$ satisfies:*

$$\forall \alpha < \aleph_1,\ 0_\alpha \in L(g) \;\&\; \theta(0_\alpha, \alpha, g).$$

PROOF. As in Propositions II–3 and II–4.

The following lemma is well known.

LEMMA III–4. *Let $\varphi(x)$ be a formula where x is a real. φ is equivalent to a π_2^1 formula if and only if there is a π_1^{ZF} formula ψ such that:*

$$\forall x(\varphi(x) \leftrightarrow H_{\aleph_1} \models \psi(x))$$

(where $H_{\aleph_1}$ is the set of the hereditary countable sets).

It is then enough to find a π_1^{ZF} formula which satisfies the conditions of the theorem.

Let us now modify — just a little — the definition of the sequence $(a_\alpha, 0_\alpha)_{\alpha<\xi}$ in Proposition I–1. We had written: "a_α is the first real x such that $L(x) = L(0_\alpha)$" but this is a Σ_2^{ZF} formula and it is too much.

LEMMA III–5. *There is a Σ_1^{ZF} formula $H(x,y,\alpha)$ such that:*

$$\mathrm{ZF} \vdash \forall y, \alpha(y \subset \alpha \;\&\; \alpha < \aleph_1^{L(y)} \rightarrow \exists! x\, H(x,y,\alpha)) \;\&$$

$$\forall x(H(x,y,\alpha) \rightarrow L(x) = L(y) \;\&\; x \subset \omega).$$

PROOF. H says: x_0 is the first real in the order of $L(y)$ which codes α and f is an isomorphism from x_0 onto α and $\forall n(n \in x_1 \leftrightarrow f(n) \in y)$ and $x = (x_0, x_1)$.

DEFINITION III–6. We modify Proposition I–1 by: "$\cdots a_\alpha$ is the unique real x such that $H(x, 0_\alpha, \omega \cdot \alpha)$. $\cdots$".

It is clear that it does not modify L^* and the propositions proved for L^*, since we have only used the fact that $L(a_\alpha) = L(0_\alpha)$.

We now give the π_1^{ZF} formula for the theorem. φ will be a formula "φ_1 & φ_2".

DEFINITION. Let $\varphi_1(x)$ be the formula:

$$x \subset \omega \ \&\ \forall \alpha < \aleph_1([\forall y, z, t, u(\theta(z, \alpha+1, x)\ \&$$

$$\theta(y, \alpha, x)\ \&\ H(t, y, \omega . \alpha))\ \&$$

$$\forall n \in \omega(n \in u \leftrightarrow \omega . \alpha + n \in z) \rightarrow u = t^*]\ \&$$

$$[\alpha = \bigcup \alpha \rightarrow \forall y(\theta(y, \alpha, x) \rightarrow (\forall z \in y\ \exists \beta < \alpha\ \forall t(\theta(t, \beta, x) \rightarrow z \in t)$$

$$\&\ \forall \beta < \alpha\ \forall t, z(\theta(t, \beta, x)\ \&\ z \in t \rightarrow z \in y)))]).$$

LEMMA III-7. (1) *φ_1 is a π_1^{ZF} formula.*
(2) *$M(g)$ satisfies*: $\varphi_1(g)\ \&\ \forall x(\varphi_1(x) \rightarrow \forall y\ \forall \alpha < \aleph_1(\theta(y, \alpha, x) \rightarrow y = 0_\alpha))$.

PROOF. (1) The formulas $\alpha < \aleph_1$, θ and H are Σ_1^{ZF}, the formula "$u = t^*$" is π_1^{ZF}.

(2) The proof is done by induction and is clear from the lemmas before.

DEFINITION. Let $\varphi_2(x)$ be the formula: $\forall \alpha < \aleph_1\ \forall T\ \forall f(T = \Pi_{n\in\omega} T'_n \restriction \alpha + 1)$ where $T'_n \restriction \alpha + 1$ is the tree constructed (in §III) in $L(y)$ with y such that $\theta(y, \alpha, x)$ and $f = (f_n)_{n\in\omega}$ where the f_n are the associated functions $\rightarrow \exists p \in T$ $\forall n \in \omega(|p_n| = \alpha \& p_n \geqq_{T'_n}$ the n-th element of $x \& \forall u(u <_{T'_{n+1}} p_{n+1}$ $\rightarrow f_{n+1}(u) \leqq_{T'_n} p_n))$.

LEMMA III–8. (1) *φ_2 is a π_1^{ZF} formula.*
(2) *$M(g)$ satisfies*: $\varphi_2(g)\ \&\ \exists! x(\varphi_1(x)\ \&\ \varphi_2(x))$.

PROOF. (1) The proof is the same as in [3]. It is enough to see that "$T = \Pi_{n\in\omega} T'_n \restriction \alpha + 1$" and "$f = (f_n)_{n\in\omega}$" are Σ_1^{ZF} formulas. It is because the construction is done by a Δ_1^{ZF} induction in $L_{\aleph_1}(y)$ with the y such that $\theta(y, \alpha, x)$.

(2) Remark that if $\varphi_1(x)$ is true then the sequence $(T'_n)_{n\in\omega}$ is exactly the good sequence $(T_n)_{n\in\omega}$ (because of Lemma III–7). Now see the proof in [3].

LEMMA III–9. *The formula* $\varphi(x): \varphi_1(x)$ & $\varphi_2(x)$ *satisfies the properties of Theorem* I.

The proof is now complete.

IV. We now give a sketch of the proof of the following theorem which gives a similar property for L^* of the one given in [2] for L.

THEOREM II. *There is a* π_3^1 *formula* $\psi(x)$ *such that if* ZF* *is consistent then so is*: $ZF^* + GCH + \aleph_1 = \aleph_1^{L^*} + \exists! x\psi(x) + \exists a \subset \omega \quad (a \notin L^* \quad \& \quad \psi(a) \quad \& \quad V = L^*(a))$.

PROOF. Let M be a model of $ZF + V = L^*$.

We define in M, as in [2], a family $(P_\alpha)_{\alpha<\aleph_1}$ of ordered sets of conditions and $P = \bigcup P_\alpha$. The only difference with [2] is that $P_{\alpha+1}$ is constructed by forcing with the same set of conditions but over $L(0_\alpha)$ instead of some L_γ; we may take the generic in $L(0_{\alpha+1})$.

It is easily seen that for all α in $\aleph_1$, P_α is in $L(0_\alpha)$.

The following lemmas will establish the theorem. The proofs are the same as in [2]; we only have to put $L(0_\alpha)$ instead of L_{γ_α}.

LEMMA IV–1. (1) P^n *satisfies the* $\aleph_1$ *chain condition.*

(2) *If a and b are P generic over M and* $a \neq b$ *then* (a, b) *is* P^2 *generic over M.*

(3) *If a is P generic over M then a is* P_α *generic over* $L(0_\alpha)$ *for all* $\alpha < \aleph_1$.

LEMMA IV–2. *Let N be an extension of M; the set* $A = \{a \subset \omega \mid a \text{ is } P \text{ generic over } L^*\}$ *is* π_3^1 *in N.*

PROOF. The formula "$x \in L^*$" is Σ_2^{ZF}. There is a Σ_2^{ZF} formula which defines a well ordering relation on L^*. We conclude with the same argument as in [2].

LEMMA IV–3. *Let g be a P generic over M. Then* $M(g)$ *satisfies* ZF*.

PROOF. Let $x \subset \omega$ be in $M(g)$. Since M satisfies $ZF + V = L^*$ there is α in $\aleph_1$ such that: $x \in L(A \cap \omega.\alpha, g) = L(a_\alpha, g)$. g is generic over $L(a_\alpha)$ (by Lemma IV–1) and $M(g)$ satisfies: a_α^* exists (it is $a_{\alpha+1}$); so it also satisfies $(a_\alpha, g)^*$ exists; and we conclude by Lemma I–4.

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UNIVERSITÉ TOULOUSE LE MIRAIL
109 RUE VAUQUELIN
31081 TOULOUSE CEDEX, FRANCE